

Irredundant and minimal covers of finite groups

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Abstract

A cover of a finite non-cyclic group G is a family \mathcal{H} of proper subgroups of G whose union equals G . A cover of G is called minimal if it has minimal size, and irredundant if it does not properly contain any other cover. We classify the finite non-cyclic groups all of whose irredundant covers are minimal.

1 Introduction

Let G be a finite non-cyclic group. A cover of G is a family \mathcal{H} of proper subgroups of G such that $G = \bigcup_{H \in \mathcal{H}} H$. Covers of groups have been studied by many authors. There are at least two interesting notions of *minimality* for a cover: size-wise or inclusion-wise. Let \mathcal{H} be a cover of G . \mathcal{H} is said to be minimal (size-wise minimal) if it has minimum size among all covers of G . \mathcal{H} is said to be irredundant (or inclusion-wise minimal) if no proper subfamily of \mathcal{H} is a cover of G . Obviously, a minimal cover is always irredundant but the converse is not always true: for instance if $n > 2$ and p is a prime, the non-trivial cyclic subgroups of the elementary abelian group C_p^n form an irredundant cover (they pairwise intersect in the identity) but only $p+1$ proper subgroups are needed to cover the group: just lift a cover of any epimorphic image of the form $C_p \times C_p$.

It is natural to ask when these two minimality notions collapse, that is, to ask when are all irredundant covers size-wise minimal. In other words, we consider the finite non-cyclic groups all of whose irredundant covers have the same size, in short we will say that they “admit only one-sized covers”. Such groups were considered by J. R. Rogério in [5]. Note that if $N \trianglelefteq G$ and \mathcal{H} is an irredundant cover of G/N then the family of preimages of the members of \mathcal{H} via the projection $G \rightarrow G/N$ is an irredundant cover of G , therefore if G admits only one-sized covers and $N \trianglelefteq G$ with G/N non-cyclic then G/N admits only one-sized covers. J. R. Rogério in [5] proved that a nilpotent group which admits only one-sized covers is necessarily of the form $H \times C$ where $|H|$ and $|C|$ are coprime, C is cyclic and H is either the quaternion group Q_8 or $C_p \times C_p$ for p a prime, and that a solvable group which admits only one-sized covers is supersolvable. In this paper we give a complete classification of groups with

only one-sized covers. The only strong tools we use for this classification are illustrated in Section 2.

The following is our main result.

Theorem 1. *Let G be a finite non-cyclic group. Then G admits only one-sized covers if and only if $G = H \times C$ where C is a cyclic group of order coprime to $|H|$ and one of the following facts happens.*

1. $H \cong C_p \times C_p$ for some prime p ,
2. $H \cong Q_8$,
3. $H \cong C_p \rtimes C_n$ where p is a prime that does not divide n and the action of C_n on C_p is non-trivial.

Let G admit only one-sized covers, and write $G = H \times C$ as in the theorem. Any irredundant cover of G has the form $\{K \times C : K \in \mathcal{H}\}$ where \mathcal{H} is an irredundant cover of H . As Rogério observed, in a nilpotent group with only one-sized covers the family of maximal cyclic subgroups is the unique irredundant cover. In the non-nilpotent case $H \cong C_p \rtimes C_n$ (case (3) of the theorem) every irredundant cover of H contains all the p -complements (there are p of them, they are cyclic of order n) and any proper subgroup containing the centralizer of the Sylow p -subgroup.

2 The main tools

Let us illustrate the known results concerning covers that we will need in our proof. Let $\sigma(G)$ be the minimal size of a cover of the finite group G , where we put $\sigma(G) = \infty$ if G is cyclic. Note that if $N \trianglelefteq G$ then $\sigma(G) \leq \sigma(G/N)$, since a cover of G/N can always be lifted to a cover of G . The number $\sigma(G)$ was computed by Tomkinson in [2] for solvable groups. He proved the following.

Theorem 2. *Let G be a solvable non-cyclic group. Then $\sigma(G)$ equals $q + 1$ where q is the smallest order of a chief factor of G with multiple complements.*

The following is a corollary of the main result of Bryce and Serena in [1].

Theorem 3. *Let G be a non-cyclic group. If G admits a cover of size $\sigma(G)$ consisting of abelian subgroups then G is solvable.*

3 Proof of Theorem 1

We start with some lemmas. Recall that a “maximal cyclic subgroup” of G is a cyclic subgroup of G which is not properly contained in any other cyclic subgroup of G . Define $\lambda(G)$ to be the maximum size of an irredundant cover of G .

Lemma 4. *Let \mathcal{C} be the family of all maximal cyclic subgroups of a finite non-cyclic group G . Then*

1. \mathcal{C} is an irredundant cover of G .
2. For any irredundant cover \mathcal{H} of G , $|\mathcal{H}| \leq |\mathcal{C}|$.
3. Let \mathcal{H} be an irredundant cover of G . Then $|\mathcal{H}| = |\mathcal{C}|$ if and only if each $H \in \mathcal{H}$ contains exactly one maximal cyclic subgroup of G .

In particular $\lambda(G)$ equals the number of maximal cyclic subgroups of G .

Now assume that G admits only one-sized covers. Then the following hold.

4. $\lambda(G) = \sigma(G)$.
5. If $N \trianglelefteq G$ and G/N is non-cyclic then $\sigma(G) = \sigma(G/N)$ and G/N has precisely $\sigma(G)$ maximal cyclic subgroups.
6. If C_1, C_2 are two distinct maximal cyclic subgroups of G , $\langle C_1 \cup C_2 \rangle = G$.

Proof. (1) is clear. To prove (2) let \mathcal{H} be an irredundant cover of G , call $k := |\mathcal{C}|$ and let x_1, \dots, x_k be fixed generators of the distinct maximal cyclic subgroups of G . Since \mathcal{H} is a cover of G , for $i = 1, \dots, k$ we can choose $H_i \in \mathcal{H}$ such that $x_i \in H_i$ (where it can happen that $H_i = H_j$ for $i \neq j$). Since any element of G is contained in a maximal cyclic subgroup of G , any element of G is a power of some x_i , hence $\{H_1, \dots, H_k\}$ is a cover of G contained in \mathcal{H} . Since \mathcal{H} is an irredundant cover $\mathcal{H} = \{H_1, \dots, H_k\}$ hence $|\mathcal{H}| \leq k = |\mathcal{C}|$. This proves (2). If equality holds, i.e. $|\mathcal{H}| = k$, then the H_i 's are pairwise distinct hence each H_i contains exactly one maximal cyclic subgroup of G . This proves (3). (4) follows from the fact that covers of minimal size are clearly irredundant, hence they have size $\lambda(G)$ as G admits only one-sized covers. We prove (5). Since G/N is non-cyclic, the cover \mathcal{C} of G consisting of the maximal cyclic subgroups maps via the projection $G \rightarrow G/N$ to a cover of G/N of the same size consisting of cyclic subgroups. Hence if G admits only one-sized covers then $\sigma(G) = |\mathcal{C}| \geq \sigma(G/N)$. Since $\sigma(G) \leq \sigma(G/N)$ we get equality: $\sigma(G) = \sigma(G/N)$. Since G admits only one-sized covers, so does G/N hence by (1) it has $\sigma(G/N) = \sigma(G)$ maximal cyclic subgroups. We prove (6). Let C_1, C_2, \dots, C_k be the maximal cyclic subgroups of G . If $\langle C_1 \cup C_2 \rangle \neq G$ then $\{\langle C_1 \cup C_2 \rangle, C_3, \dots, C_k\}$ would be a cover of G of size $k - 1 = \lambda(G) - 1$, contradicting (4). \square

Lemma 5. *Let G be a finite solvable group and let p be a prime divisor of $|G|$. Assume that all complemented chief factors of G which are p -groups are central. Then G is p -nilpotent.*

Proof. By induction on $|G|$. Let N be a minimal normal subgroup of G . Then G/N is p -nilpotent by induction. Let $\Phi(G)$ denote the Frattini subgroup of G . If $N \subseteq \Phi(G)$ then G is p -nilpotent by [4], Lemma 13.2 in Chapter A, so now assume N is non-Frattini, i.e. N is complemented. If N is a p -group then by hypothesis N is central hence $G = H \times N$ with H p -nilpotent, hence G is p -nilpotent. Now assume N is not a p -group. Let K/N be a normal p -complement of G/N . Since p does not divide $|N|$, p does not divide $|K|$ hence K is a normal p -complement of G thus G is p -nilpotent. \square

Lemma 6. *Let G be a group with a maximal core-free cyclic subgroup H and a normal subgroup N complemented by H . Then N with the conjugates of H form an irredundant cover of G of size $|N| + 1$.*

Proof. If K is a conjugate of H different from H then $H \cap K$ is normal in H and K because H and K are cyclic, and $\langle H, K \rangle = G$ because H, K are maximal subgroups of G . Therefore $H \cap K \trianglelefteq G$. Since H is core-free, $H \cap K = \{1\}$. It follows that N with the conjugates of H cover exactly

$$|N| + (|H| - 1)|N| = |N||H| = |G|$$

elements, hence they form a cover \mathcal{H} of G which is irredundant because any two members of \mathcal{H} intersect in $\{1\}$. \square

Now we proceed with the proof of Theorem 1.

Let G admit only one-sized covers. Let \mathcal{C} be the family of maximal cyclic subgroups of G . From Lemma 4 it follows that $|\mathcal{C}| = \sigma(G)$. It follows by Theorem 3 that G is solvable. We prove that G is supersolvable. By [3] 5.4.7 and 9.4.4 a finite group is supersolvable if and only if all of its maximal subgroups have prime index. Let M be a maximal subgroup of G , and let $|G : M| = q = p^k$ a prime power. We need to show that q is a prime, i.e. $k = 1$. Let $X := G/M_G$ where M_G is the normal core of M in G . X is a primitive solvable group of degree q , hence $X = V \rtimes H$ with $|V| = q$ and H an irreducible subgroup of $GL(V)$. If $H = \{1\}$ then $q = |V|$ is a prime. Now assume $H \neq \{1\}$, so that X is non-nilpotent. Since G admits only one-sized covers, so does X by Lemma 4 (5). Let ℓ be a prime dividing $|H|$ but not $|V|$. Let $h \in H$ of order ℓ and let $\langle x \rangle$ a maximal cyclic subgroup of X containing h . Note that if $v \in V$ normalizes $\langle h \rangle$ then $[v, h] = v h v^{-1} h^{-1} \in H \cap V = \{1\}$ hence v centralizes h . Since $C_H(V) = \{1\}$ it follows that there exists $v \in V$ with $\langle h \rangle^v \neq \langle h \rangle$. In particular $\langle x \rangle \neq \langle x^v \rangle$ hence $\langle x \rangle$ and $\langle x^v \rangle$ are two distinct maximal cyclic subgroups of X . Therefore by Lemma 4 (6) since X admits only one-sized covers $X = \langle x, x^v \rangle \subseteq V \langle x \rangle \subseteq X$ hence $X = V \langle x \rangle$ and hence H is cyclic (actually $H = \langle x \rangle$). By Lemma 6 it follows that X has an irredundant cover of size $q + 1$, hence $\lambda(X) = \sigma(X) = q + 1$. If V was not cyclic then writing V as union of cyclic subgroups intersecting in the identity would give, with the q conjugates of H , an irredundant cover of X of size larger than $q + 1$. Hence V is cyclic, i.e. $q = p$.

We deal with the nilpotent case and with the supersolvable non-nilpotent case separately.

First, let G be nilpotent. Let P be a Sylow p -subgroup of G . Then P is an epimorphic image of G hence if P is non-cyclic, by Lemma 4 (5) $\sigma(G) = \sigma(P) = p + 1$. This means that exactly one of the Sylow subgroups of G is non-cyclic, hence G has the form $P \times H$ with H a cyclic Hall direct factor and P a Sylow p -subgroup. Hence we may assume that $G = P$, i.e. G is a p -group. We show that all maximal subgroups of G are cyclic. Let M_1, \dots, M_k be the maximal subgroups of G . By Lemma 4(6) G is 2-generated hence $G/\Phi(G) \cong C_p \times C_p$,

so that $k = p + 1$. Since G has exactly $p + 1$ maximal cyclic subgroups, we are left to show that all maximal cyclic subgroups of G are maximal subgroups. By Lemma 4 (3) each M_i contains exactly one maximal cyclic subgroup of G . Let C be a maximal cyclic subgroup of G . Without loss of generality $C \subseteq M_1$. Then $C \cup M_2 \cup \dots \cup M_k = G$ hence setting $c := |C|$ since $|\Phi(G)| = p^{n-2}$ we must have

$$p(p^{n-1} - p^{n-2}) + p^{n-2} + c > p^n,$$

and this implies $c > p^{n-2}(p-1)$. Since c is a power of p we deduce $c = p^{n-1}$, i.e. C is a maximal subgroup of G . This proves that all maximal subgroups of G are cyclic. Let now x be a central element of G of order p . If G has an element y of order p which is not a power of x then $G \geq \langle x \rangle \langle y \rangle \cong C_p \times C_p$. If $\langle x \rangle \langle y \rangle$ equals G then $G \cong C_p \times C_p$. Suppose this is not the case. Then G has a proper non-cyclic subgroup $\langle x \rangle \langle y \rangle$, and this contradicts the fact that all maximal subgroups of G are cyclic. Thus $\langle x \rangle$ is the unique subgroup of G of order p , hence by [3] 5.3.6 G is a generalized quaternion group. Let N be the unique subgroup of G of order 2. Then G/N is both a dihedral 2-group and a 2-group which admits only one-sized covers, hence by what we have proved either $G \cong C_2 \times C_2$ or G/N is a generalized quaternion group. Therefore $G/N \cong C_2 \times C_2$, i.e. $G \cong Q_8$.

Let now G be supersolvable and non-nilpotent. Since G is non-nilpotent there exists a non-central complemented chief factor C of G , with $|C| = p$. Write $C = H/K$ and let A/K be a complement of H/K in G/K . Let $X := (G/K)/(C_{A/K}(H/K))$. Then X is a primitive group and

$$(H/K)(C_{A/K}(H/K))/(C_{A/K}(H/K)) \cong C$$

is its unique minimal normal subgroup (which we will call again C), which is complemented by $(A/K)/(C_{A/K}(H/K))$. Since C has prime order, the complements of C in X are cyclic. By Lemma 4(5) and Lemma 6 it follows that $\sigma(G) = \sigma(X) = p + 1$. Therefore all non-central complemented chief factors of G have the same order p . It follows that if a complemented chief factor of G has order $q \neq p$ then it is central. By Lemma 5 it follows that G is q -nilpotent for all prime divisors q of $|G|$ different from p . The intersection of the normal q -complements for $q \neq p$ is then equal to the unique Sylow p -subgroup of G , call it P . By the Schur-Zassenhaus theorem there is a complement H of P , hence $G = P \rtimes H$.

If H was not cyclic then $p + 1 = \sigma(G) = \sigma(H)$, contradicting Theorem 2 because H is solvable and $|H|$ is coprime to p . Hence $H = \langle a \rangle$ is cyclic. Since G is non-nilpotent H does not centralize P . Let $F := \Phi(P)$. By [3] 5.3.3 H does not centralize P/F . Hence G/F admits a non-cyclic quotient of the form $C_p \rtimes C_n$, being n a positive integer, and by Lemma 4 (5) and Lemma 6 $\sigma(G) = \sigma(G/F) = p + 1$. Write $P/F = C_p^t$. The cyclic subgroups of G/F have order at most pn , on the other hand G/F is the union of $p + 1$ cyclic subgroups hence $(p + 1)(pn - 1) + 1 \geq p^t n$ and we deduce $t \leq 2$. Suppose $t = 2$. G/F must have central elements of order p otherwise the cyclic subgroups would have order n or p and they couldn't cover G/F ; but then $G/F \cong C_p \times H$ with

$H = C_p \rtimes C_n$ non-abelian: this group has more than $p + 1$ maximal cyclic subgroups. Therefore $t = 1$, i.e. P is cyclic, say $P \cong C_{p^k}$.

We are left to prove that $k = 1$. Up to quotienting out by $C_H(P)$ we may assume that $C_H(P) = \{1\}$ hence P is a maximal cyclic subgroup of G . Let $A := C_P(H)$. Then AH is a maximal cyclic subgroup of G and it has at least p conjugates because AH is not normal in G being H not normal in G . Also, every conjugate of AH contains A . Since G has exactly $p + 1$ maximal cyclic subgroups which form a cover we must then have

$$p(|AH| - |A|) + |P| \geq |G| = |P||H|,$$

and since $A \neq P$ we deduce $|P : A| = p$. We must show that $A = \{1\}$ (this is equivalent to $k = 1$). $H = \langle a \rangle$ acts on $P = \langle x \rangle$ by raising x to a power l . Since H centralizes $A = \langle x^p \rangle$ we have $x^p = a^{-1}x^p a = x^{pl}$ hence $pl \equiv p \pmod{p^k}$, i.e. p^{k-1} divides $l - 1$, in particular l is coprime to p . On the other hand since H acts faithfully (because $C_H(P) = \{1\}$) l divides $\varphi(p^k) = p^{k-1}(p - 1)$, so since $(l, p) = 1$, l divides $p - 1$. But then $p^{k-1} \leq l \leq p - 1$ which forces $k = 1$.

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